

**JEE ADVANCED**

**Single Correct Answer Type**

1. The points  $(-a, -b)$ ,  $(0, 0)$ ,  $(a, b)$  and  $(a^2, ab)$  are
  - a. collinear
  - b. vertices of a parallelogram
  - c. vertices of a rectangle
  - d. none of these

**(IIT-JEE 1979)**
2. The point  $(4, 1)$  undergoes the following three transformations successively.
  - i. Reflection about the line  $y = x$ .
  - ii. Translation through a distance 2 units along the positive direction of the  $x$ -axis.
  - iii. Rotation through an angle  $\pi/4$  about the origin in the counterclockwise direction.

Then the final position of the point is given by the coordinates

a. $(1/\sqrt{2}, 7/\sqrt{2})$	b. $(-\sqrt{2}, 7\sqrt{2})$
c. $(-1/\sqrt{2}, 7/\sqrt{2})$	d. $(\sqrt{2}, 7\sqrt{2})$

**(IIT-JEE 1980)**

3. If  $x_1, x_2, x_3$  as well as  $y_1, y_2, y_3$  are in GP with the same common ratio, then the points  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ 
  - a. lie on a straight line
  - b. lie on an ellipse
  - c. lie on a circle
  - d. are the vertices of a triangle

**(IIT-JEE 1999)**
4. The incenter of the triangle with vertices  $(1, \sqrt{3})$ ,  $(0, 0)$ , and  $(2, 0)$  is
 

a. $(1, \sqrt{3}/2)$	b. $(2/3, 1/\sqrt{3})$
c. $(2/3, \sqrt{3}/2)$	d. $(1, 1/\sqrt{3})$

**(IIT-JEE 2000)**
5. Let  $0 < \alpha < \pi/2$  be a fixed angle. If  $P \equiv (\cos \theta, \sin \theta)$  and  $Q \equiv (\cos(\alpha - \theta), \sin(\alpha - \theta))$ , then  $Q$  is obtained from  $P$  by the
  - a. clockwise rotation around the origin through an angle  $\alpha$
  - b. anticlockwise rotation around the origin through an angle  $\alpha$



- c. reflection in the line through the origin with slope  $\tan \alpha$   
d. reflection in the line through the origin with slope  $\tan(\alpha/2)$  (IIT-JEE 2002)
6. The number of integral points (integral point means both the coordinates should be integers) exactly in the interior of the triangle with vertices  $(0, 0)$ ,  $(0, 21)$ , and  $(21, 0)$  is  
a. 133    b. 190    c. 233    d. 105 (IIT-JEE 2003)
7. The orthocenter of the triangle with vertices  $(0, 0)$ ,  $(3, 4)$ , and  $(4, 0)$  is  
a.  $(3, 5/4)$     b.  $(3, 12)$     c.  $(3, 3/4)$     d.  $(3, 9)$  (IIT-JEE 2003)
8. Let  $O(0, 0)$ ,  $P(3, 4)$ , and  $Q(6, 0)$  be the vertices of triangle  $OPQ$ . The point  $R$  inside the triangle  $OPQ$  is such that the triangles  $OPR$ ,  $PQR$ ,  $OQR$  are of equal area. The coordinates of  $R$  are  
a.  $(4/3, 3)$     b.  $(3, 2/3)$     c.  $(3, 4/3)$     d.  $(4/3, 2/3)$  (IIT-JEE 2007)
9. Consider three points  $P \equiv (-\sin(\beta - \alpha), -\cos \beta)$ ,  $Q \equiv (\cos(\beta - \alpha), \sin \beta)$ , and  $R \equiv (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$ , where  $0 < \alpha, \beta, \theta < \pi/4$ . Then  
a.  $P$  lies on the line segment  $RQ$   
b.  $Q$  lies on the line segment  $PR$   
c.  $R$  lies on the line segment  $QP$   
d.  $P, Q, R$  are non-collinear (IIT-JEE 2008)

### Multiple Correct Answers Type

1. If the vertices  $P, Q$ , and  $R$  of a triangle  $PQR$  are rational points, which of the following points of triangle  $PQR$  is (are) always rational point(s)? (A rational point is a point whose coordinates are rational numbers.)  
a. Centroid    b. Incenter  
c. Circumcenter    d. Orthocenter (IIT-JEE 1982)
2. The points  $(0, 8/3)$ ,  $(1, 3)$ , and  $(82, 30)$  are the vertices of  
a. an obtuse-angled triangle  
b. an acute-angled triangle  
c. a right-angled triangle  
d. none of these (IIT-JEE 1986)

3. If  $P(1, 2)$ ,  $Q(4, 6)$ ,  $R(5, 7)$ , and  $S(a, b)$  are the vertices of a parallelogram  $PQRS$ , then  
a.  $a = 2, b = 4$     b.  $a = 3, b = 4$   
c.  $a = 2, b = 3$     d.  $a = 1$  or  $b = -1$  (IIT-JEE 1998)

### Subjective Type

1. The area of a triangle is 5. Two of its vertices are  $A(2, 1)$  and  $B(3, -2)$ . The third vertex  $C$  is on  $y = x + 3$ . Find  $C$ . (IIT-JEE 1978)
2. A straight line segment of length  $l$  moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line segment in the ratio  $1 : 2$ . (IIT-JEE 1978)
3. Two vertices of a triangle are  $(5, -1)$  and  $(-2, 3)$ . If the orthocenter of the triangle is the origin, find the coordinates of the third point. (IIT-JEE 1979)
4. The vertices of a triangle are  $(at_1t_2, a(t_1 + t_2))$ ,  $(at_2t_3, a(t_2 + t_3))$ , and  $(at_3t_1, a(t_3 + t_1))$ . Find the orthocenter of the triangle. (IIT-JEE 1983)
5. The coordinates of  $A, B, C$  are  $(6, 3)$ ,  $(-3, 5)$ ,  $(4, -2)$ , respectively, and  $P$  is any point  $(x, y)$ . Show that the ratio of the area of  $\Delta PBC$  to that of  $\Delta ABC$  is  $|x + y - 2|/7$ . (IIT-JEE 1983)
6. A line cuts the  $x$ -axis at  $A(7, 0)$  and the  $y$ -axis at  $B(0, -5)$ . A variable line  $PQ$  is drawn perpendicular to  $AB$  cutting the  $x$ -axis at  $P$  and the  $y$ -axis at  $Q$ . If  $AQ$  and  $BP$  intersect at  $R$ , then find the locus of  $R$ . (IIT-JEE 1990)
7. For points  $P \equiv (x_1, y_1)$  and  $Q \equiv (x_2, y_2)$  of the coordinate plane, a new distance  $d(P, Q)$  is defined by  $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$ . Let  $O = (0, 0)$  and  $A = (3, 2)$ . Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from  $O$  and  $A$  consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram. (IIT-JEE 2000)

### Answer Key

#### JEE Advanced

##### Single Correct Answer Type

1. a.    2. c.    3. a.    4. d.    5. d.  
6. b.    7. c.    8. c.    9. d.

##### Multiple Correct Answers Type

1. a., c., d.    2. d.    3. c.

##### Subjective Type

1.  $(7/2, 13/2)$  or  $(-3/2, 3/2)$   
3.  $(-4, -7)$   
4.  $(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$   
6.  $x^2 + y^2 - 7x + 5y = 0$



# Hints and Solutions

## JEE Advanced

### Single Correct Answer Type

1. a. We have points  $A(-a, -b)$ ,  $B(0, 0)$ ,  $C(a, b)$  and  $D(a^2, ab)$

$$\text{Slope of } AB, m_1 = \frac{b}{a}$$

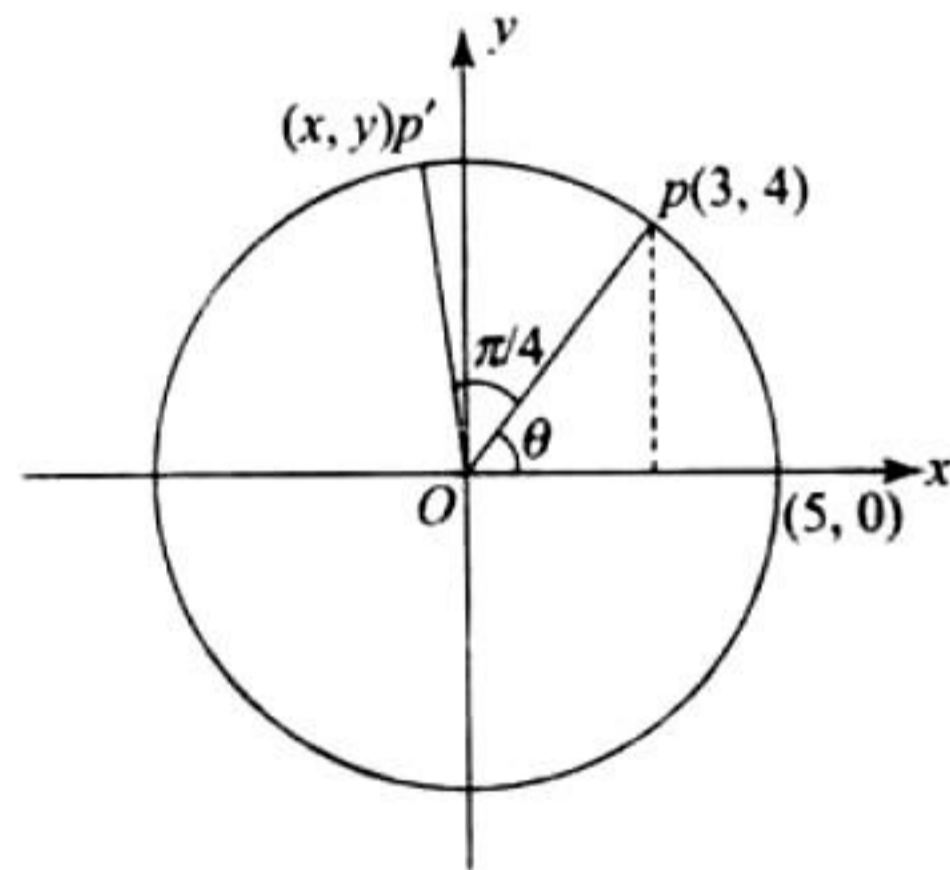
$$\text{Slope of } BC, m_2 = \frac{b}{a}$$

$$\text{Slope of } CD, m_3 = \frac{ab - b}{a^2 - a} = \frac{b(a-1)}{a(a-1)} = \frac{b}{a}$$

$$\text{Slope of } AD, m_3 = \frac{b - (-b)}{a - (-a)} = \frac{b}{a}$$

Thus points  $A, B, C, D$  are collinear.

2. c. Reflection about the line  $y = x$  changes the point  $(4, 1)$  to  $(1, 4)$ . On the translation of  $(1, 4)$  through a distance of 2 units along the positive direction of the  $x$ -axis, the point becomes  $(1 + 2, 4)$ , i.e.,  $(3, 4)$ .



On rotation about the origin through an angle  $\pi/4$ , point  $P$  takes the position  $P'$  such that  $OP = OP'$ . Also,  $OP = 5 = OP'$  and  $\cos \theta = 3/5$ ,  $\sin \theta = 4/5$ . Now,

$$x = OP' \cos\left(\frac{\pi}{4} + \theta\right)$$

$$= 5\left(\cos \frac{\pi}{4} \cos \theta - \sin \theta\right)$$

$$= 5\left(\frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}$$

$$y = OP' \sin\left(\frac{\pi}{4} + \theta\right)$$

$$= 5\left(\sin \frac{\pi}{4} \cos \theta + \cos \frac{\pi}{4} \sin \theta\right)$$

$$= 5\left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}\right) = \frac{7}{\sqrt{2}}$$

$$\therefore P' \equiv \left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

3. a. Let  $x_2 = x_1 r$ ,  $x_3 = x_1 r^2$  and so is  $y_2 = y_1 r$ ,  $y_3 = y_1 r^2$ , where  $r$  is the common ratio of G.P. Therefore,

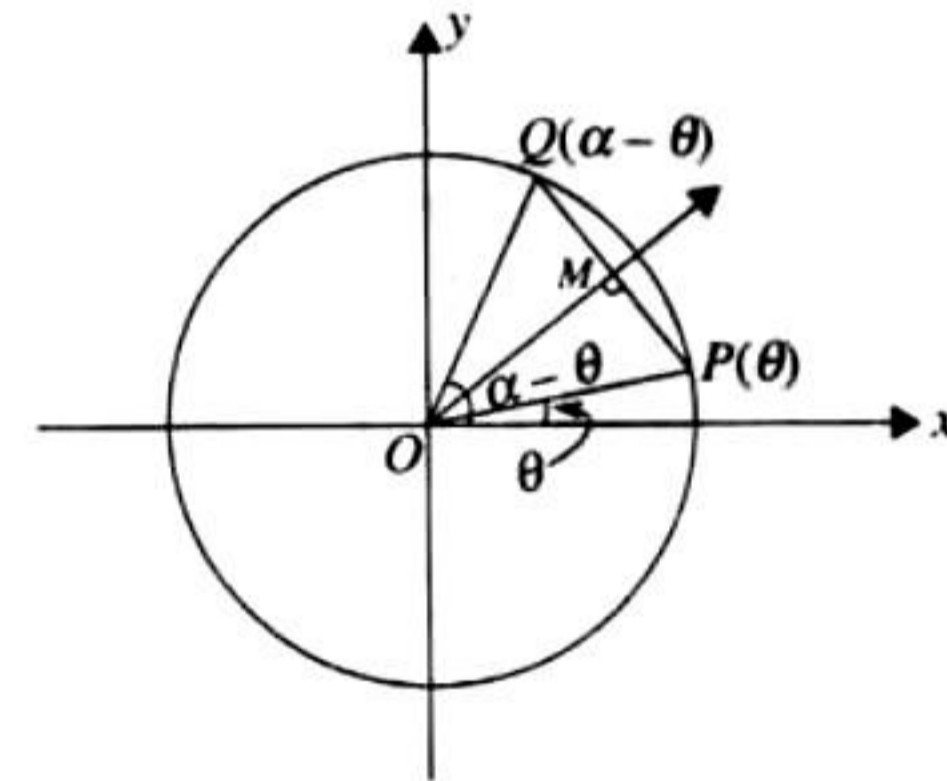
$$\begin{aligned} \Delta &= \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix} \\ &= r \times r^2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0 \end{aligned}$$

Hence, the points are collinear.

4. d. Vertices of  $\Delta$  are  $A(1, \sqrt{3})$ ,  $B(0, 0)$  and  $C(2, 0)$ . Here,  $AB = BC = CA = 2$ . So, it is an equilateral triangle and the incenter coincides with the centroid. Therefore, the centroid is

$$\left(\frac{0+1+2}{3}, \frac{0+0+\sqrt{3}}{3}\right) \equiv \left(1, \frac{1}{\sqrt{3}}\right)$$

5. d. Clearly points  $P(\cos \theta, \sin \theta)$  and  $Q(\cos(\alpha - \theta), \sin(\alpha - \theta))$  lie on circle of unit radius.



In the fig.,  $\angle POX = \theta$

$$\angle QOX = \alpha - \theta$$

$$\therefore \angle QOP = \alpha - 2\theta$$

Now  $\Delta QOP$  is isosceles.

$\therefore$  Altitude or angle bisector  $OM$  is perpendicular bisector of  $PQ$ .

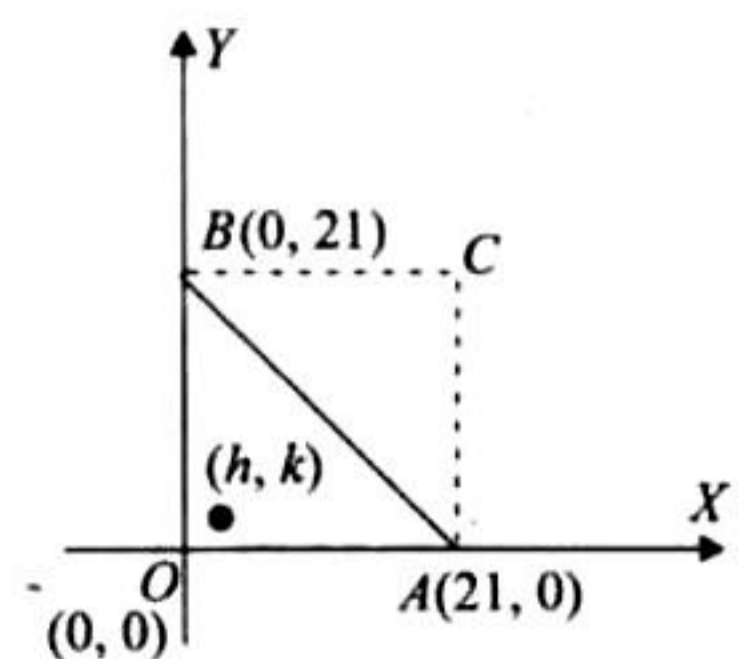
$$\therefore \angle MOP = \frac{\alpha - 2\theta}{2} = \frac{\alpha}{2} - \theta$$

$$\therefore \angle MOX = \theta + \left(\frac{\alpha}{2} - \theta\right) = \frac{\alpha}{2}$$

Thus point  $P$  is reflection of point  $Q$  in line  $OM$  having slope  $\tan \frac{\alpha}{2}$ .

6. b. The total number of points within the square  $OABC$  is  $20 \times 20 = 400$ .

The total number of points on line  $AB$ ,  $x + y = 21$  is 20, the points being  $(1, 20)$ ,  $(2, 19)$ , ...,  $(20, 1)$ . Therefore, the total number of points within  $\Delta OBC$  and  $\Delta ABC$  is  $400 - 20 = 380$ . By symmetry, the number of points within  $\Delta OAB$  is  $380/2 = 190$ .





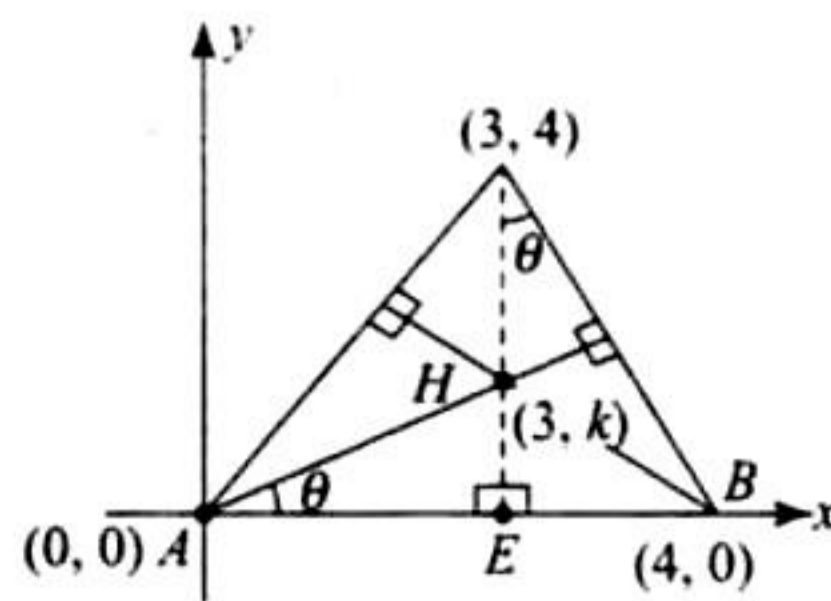
7. c. In  $\triangle AEH$ ,  $\tan \theta = \frac{k}{3}$  (i)

In  $\triangle CEB$   $\tan \theta = \frac{1}{4}$  (ii)

By (i) and (ii),  $\frac{k}{3} = \frac{1}{4}$

$\therefore k = \frac{3}{4}$

$\therefore$  Orthocenter  $\equiv \left(3, \frac{3}{4}\right)$



8. c. Centroid of triangle divides triangle into three equal area triangles

**Proof:**

Let  $G$  be centroid of triangle  $ABC$ .

Let  $AG$  meet opposite side  $BC$  in point  $D$ .

$\therefore$  Area of triangle  $ABD$  = Area of triangle  $ACD$

(as  $BD = CD$  and length of altitude from  $A$  on  $BC$  is same for both triangles)

For the same reason,

area of triangle  $GBD$  = area of triangle  $GCD$

$\therefore$  Area of triangle  $ABD$  - Area of triangle  $GBD$  = Area of triangle  $ACD$  - area of triangle  $GCD$

$\therefore$  Area of triangle  $AGB$  = Area of triangle  $AGC$

Similarly we can prove that, area of triangle  $AGC$  = Area of triangle  $BGC$

Thus area of triangle  $AGB$  = Area of triangle  $AGC$  = Area of triangle  $BGC$

Hence, centroid of triangle divides triangle into three equal area triangles.

So, in given question  $R$  is centroid of triangle  $OPQ$ , which is given by

$$R\left(\frac{3+6+0}{3}, \frac{4+0+0}{3}\right) \text{ or } R\left(3, \frac{4}{3}\right)$$

9. d.  $P \equiv (-\sin(\beta - \alpha), -\cos \beta) \equiv (x_1, y_1)$

$Q \equiv (\cos(\beta - \alpha), \sin \beta) \equiv (x_2, y_2)$

$R \equiv (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$

or  $R \equiv (x_2 \cos \theta + x_1 \sin \theta, y_2 \cos \theta + y_1 \sin \theta)$

If  $T \equiv \left(\frac{x_2 \cos \theta + x_1 \sin \theta}{\cos \theta + \sin \theta}, \frac{y_2 \cos \theta + y_1 \sin \theta}{\cos \theta + \sin \theta}\right)$ , for some ' $\theta$ '

then  $P, Q$ , and  $T$  are collinear.

$\therefore P, Q, R$  are collinear if  $\cos \theta + \sin \theta = 1$ , which is not possible as  $0 < \theta < \pi/4$ .

Hence,  $P, Q, R$  are non-collinear.

### Multiple Correct Answers Type

1. a., c., d. If  $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ , where  $x_1, y_1$ , etc., are rational numbers, then  $\Sigma x_i, \Sigma y_i$  are also rational.

So, the coordinates of the centroid  $(\Sigma x_i/3, \Sigma y_i/3)$  will be rational.

As  $AB = c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ ,  $c$  may or may not be rational and it may be an irrational number of the form  $\sqrt{p}$ . Hence, the coordinates of the incenter  $(\Sigma ax_i/\Sigma a, \Sigma ay_i/\Sigma a)$  may or may not be rational. If  $P(\alpha, \beta)$  is the circumcenter, then using  $AP = BP = CP$ , we get two equations in  $\alpha$  and  $\beta$  with rational coefficients, solving which we get the rational values of  $\alpha$  and  $\beta$ .

If  $P(\alpha, \beta)$  is the orthocenter, then using  $[(\text{slope of } AP) \times (\text{slope of } BC) = -1]$  and  $[(\text{slope of } BP) \times (\text{slope of } AC) = -1]$ , we get two equations in  $\alpha$  and  $\beta$  with rational coefficients, solving which we get the rational values of  $\alpha$  and  $\beta$ .

2. d. Let  $A \equiv (0, 8/3), B \equiv (1, 3)$ , and  $C \equiv (82, 30)$ .

Now, the slope of line  $AB$  is  $(3 - 8/3)/(1 - 0) = 1/3$ . The slope of line  $BC$  is  $(30 - 3)/(82 - 1) = 27/81 = 1/3$ . Therefore,  $AB \parallel BC$  and  $B$  is the common point. Hence,  $A, B$ , and  $C$  are collinear.

3. c.  $PQRS$  will represent a parallelogram if and only if the midpoint of  $PR$  is the same as that of the midpoint of  $QS$ . Hence,

$$\frac{1+5}{2} = \frac{4+a}{2} \text{ and } \frac{2+7}{2} = \frac{6+b}{2}$$

or  $a = 2$  and  $b = 3$ .

### Subjective Type

1. As  $C$  lies on the line  $y = x + 3$ , let the coordinates of  $C$  be  $(\lambda, \lambda + 3)$ . Also,  $A \equiv (2, 1), B \equiv (3, -2)$ . Then the area of  $\triangle ABC$  is given by

$$\frac{1}{2} \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ \lambda & \lambda + 3 & 1 \end{vmatrix} = 5$$

or  $|4\lambda - 4| = 10$

i.e.,  $4\lambda - 4 = 10$  or  $4\lambda - 4 = -10$

i.e.,  $4\lambda = 14$  or  $4\lambda = -6$

i.e.,  $\lambda = \frac{7}{2}$  or  $\lambda = -\frac{3}{2}$

Hence, the coordinates of  $C$  are  $(7/2, 13/2)$  or  $(-3/2, 3/2)$ .

2. Let  $P(x, y)$  divide line segment  $AB$  in the ratio  $1 : 2$ , so that  $AP = l/3$  and  $BP = 2l/3$  where  $AB = l$

Then  $PN = x$  and  $PM = y$   
Let  $\angle PAM = \theta = \angle BPN$

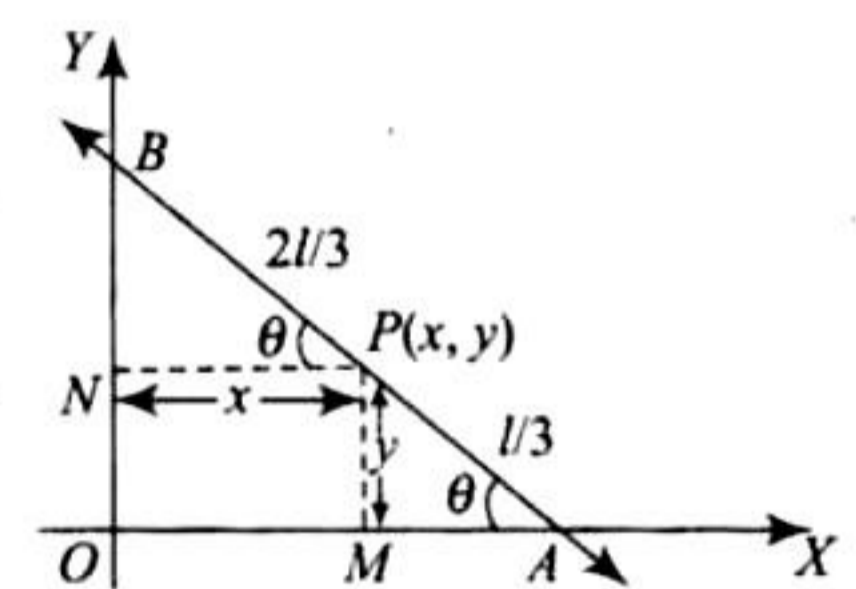
In  $\triangle PMA$ ,  $\sin \theta = \frac{y}{l/3} = \frac{3y}{l}$

In  $\triangle PNB$ ,  $\cos \theta = \frac{x}{2l/3} = \frac{3x}{2l}$

Now,  $\sin^2 \theta + \cos^2 \theta = 1$

$$\Rightarrow \frac{9y^2}{l^2} + \frac{9x^2}{4l^2} = 1$$

$$\Rightarrow 9x^2 + 36y^2 = 4l^2 \text{ which is the required locus.}$$





3. We have

$$AH \perp BC$$

$$\therefore m_{AH} \times m_{BC} = -1$$

$$\text{or } \frac{k}{h} \times \frac{3+1}{-2-5} = -1$$

$$\text{or } 4k - 7h = 0$$

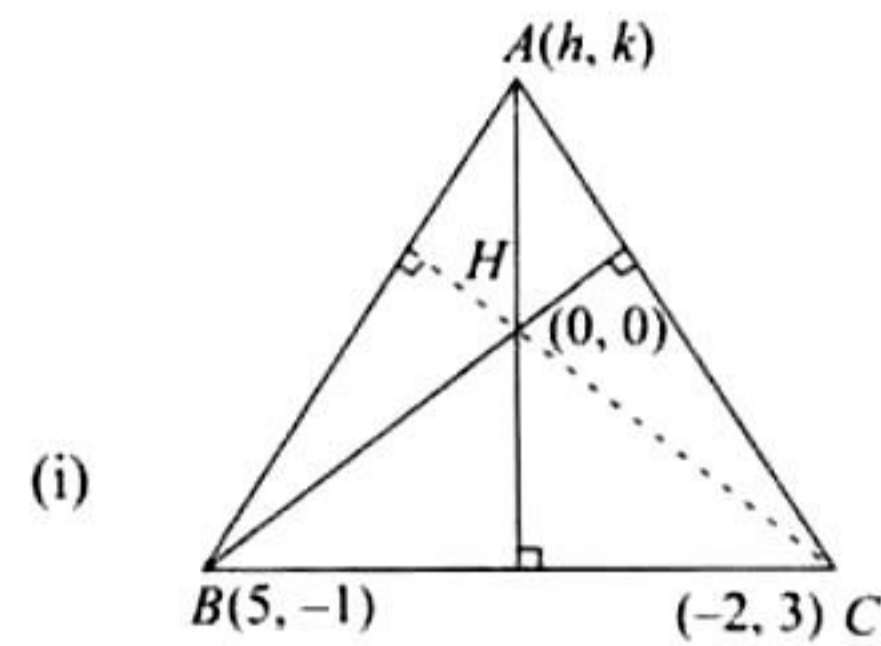
$$\text{Also, } BH \perp AC$$

$$\therefore \frac{-1}{5} \times \frac{3-k}{-2-h} = -1$$

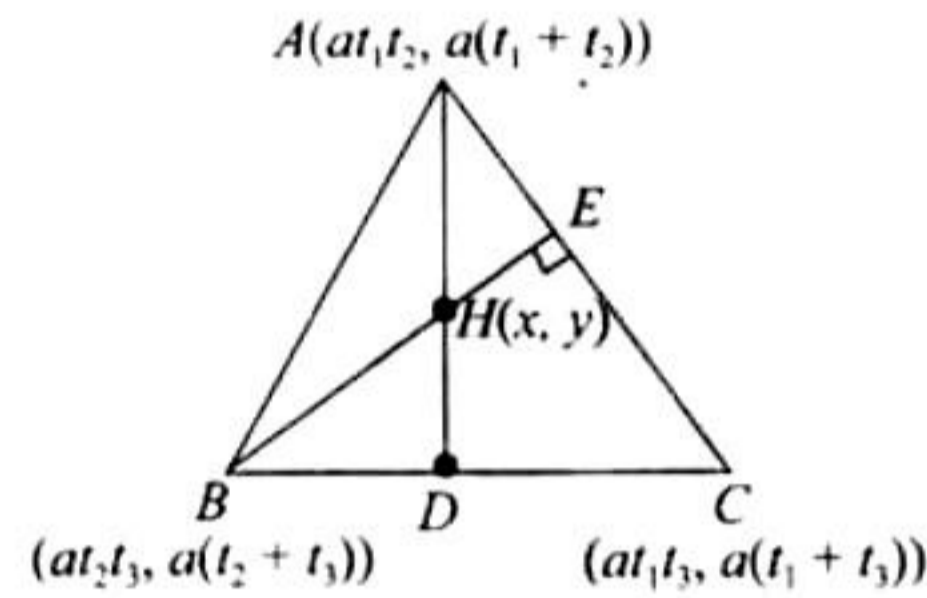
$$\text{or } 3 - k = -10 - 5h$$

$$\text{or } 5h - k + 13 = 0$$

Solving (i) and (ii), we get  $h = -4$ ,  $k = -7$ . Hence, the third vertex is  $(-4, -7)$ .



4.



We know that orthocenter of  $\Delta$  is point of concurrency of altitudes.

Let orthocenter be  $H(x, y)$ .

$$\begin{aligned} \text{Now, slope of } BC &= \frac{a(t_1 + t_3) - a(t_2 + t_3)}{at_1t_3 - at_2t_3} \\ &= \frac{a(t_1 + t_3 - t_2 - t_3)}{at_3(t_1 - t_2)} = \frac{1}{t_3} \end{aligned}$$

$$\therefore \text{Slope of } AD = -t_3$$

$$\text{Also, slope of } AD = \text{Slope of } AH = \frac{y - a(t_1 + t_2)}{x - at_1t_2} = -t_3$$

$$\therefore y - a(t_1 + t_2) = -t_3(x - at_1t_2)$$

$$\text{or } xt_3 + y = at_1t_2t_3 + a(t_1 + t_2) \quad (1)$$

Similarly, by symmetry from  $AC \perp BE$ , we get

$$xt_1 + y = at_1t_2t_3 + a(t_2 + t_3) \quad (2)$$

Solving (1) and (2) we get

$$x = -a, y = a(t_1 + t_2 + t_3) + at_1t_2t_3$$

$\therefore$  Orthocenter is  $H(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$

$$\begin{aligned} 5. \text{ Area of } \Delta ABC &= \frac{1}{2} \begin{vmatrix} 6 & 3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix} \\ &= \frac{1}{2} \times [6(7) + 3(5) + 4(-2)] = \frac{49}{2} \end{aligned}$$

$$\begin{aligned} \text{Area of } \Delta PBC &= \frac{1}{2} \begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix} \\ &= \frac{1}{2} (7x + 7y - 14) = \frac{7}{2} |x + y - 2| \end{aligned}$$

Hence, the ratio of the area of triangles is given by

$$\frac{\text{Ar}(\Delta PBC)}{\text{Ar}(\Delta ABC)} = \frac{(7/2)|x + y - 2|}{49/2} = \frac{|x + y - 2|}{7}$$

6. From the figure in  $\Delta ABQ$ ,

$$AP \perp BQ, PQ \perp AB$$

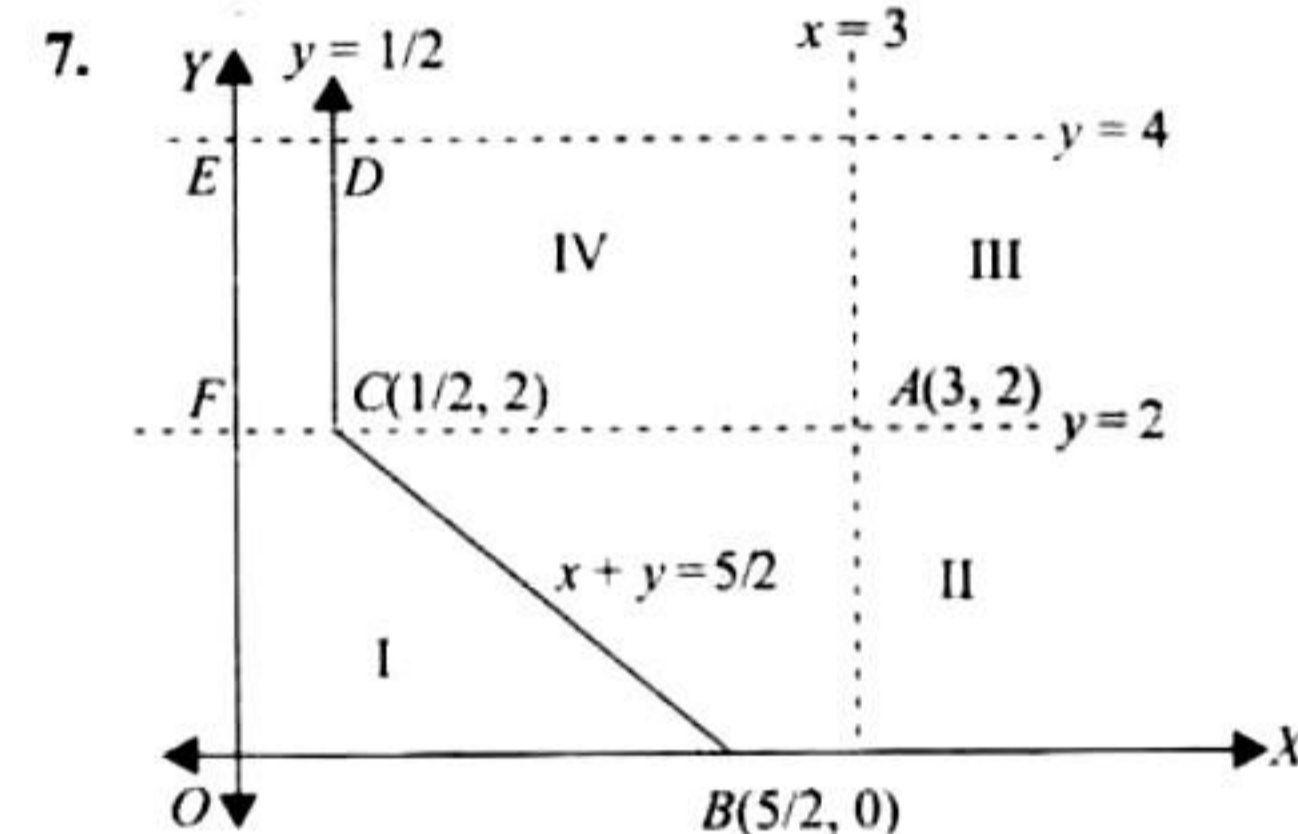
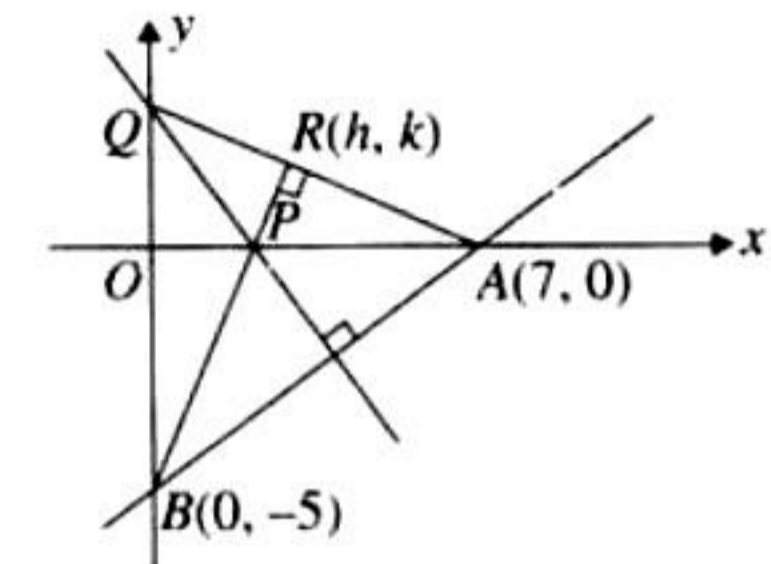
Then, we have  $BP \perp AQ$  (as the altitudes of triangle are concurrent). Hence,

$$BR \perp AR$$

$$\text{or } \frac{k-0}{h-7} \times \frac{k+5}{h-0} = -1$$

Therefore, the locus of  $R$  is

$$x^2 + y^2 - 7x + 5y = 0$$



Let  $P = (h, k)$  be a general point in the first quadrant such that  $d(P, A) = d(P, O)$ .

$$\Rightarrow |h - 3| + |k - 2| = |h| + |k| = h + k \quad (1)$$

[ $h$  and  $k$  are +ve, point  $P(h, k)$  being in I quadrant]

If  $h < 3$ ;  $k < 2$  then  $(h, k)$  lies in region I.

$$\Rightarrow 3 - h + 2 - k = h + k \Rightarrow h + k = 5/2$$

If  $h > 3$ ,  $k < 2$ ,  $(h, k)$  lies in region II.

$$\Rightarrow h - 3 + 2 - k = h + k \Rightarrow k = -1/2, \text{ which is not possible}$$

If  $h > 3$ ,  $k > 2$   $(h, k)$  lies in region III.

$$\Rightarrow h - 3 + k - 2 = h + k \Rightarrow -5 = 0, \text{ which is not possible}$$

If  $h < 3$ ,  $k > 2$   $(h, k)$  lies in region IV.

$$\Rightarrow 3 - h + k - 2 = h + k \Rightarrow h = 1/2$$

Hence required set consists of line segment  $x + y = 5/2$  of finite length as shown in the first region and the ray  $x = 1/2$  in the fourth region.